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Cooperative ring exchange in the $n = 1$ Landau level

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Abstract. We consider the semiclassical description of the fractional quantum Hall effect based on cooperative ring exchange. We generalize the original picture to the $n = 1$ Landau level and look at its validity. We find that the Wigner-crystal-like configurations which are the starting point for the picture are unstable for filling fractions $\nu_1 \geq 1/3$. For $\nu_1 < 1/3$ the effects of ring exchange are quantitatively very similar to those in the $n = 0$ Landau level.

1. Introduction

The theory of the fractional quantum Hall effect (FQHE) is based on Laughlin's wavefunction [1] and its generalizations [2]. These include hierarchical fluids, spin-singlet states and higher Landau level effects. A semiclassical account of the FQHE due to Kivelson, Kallin, Arovas and Schrieffer [3] (KKAS) has also been proposed. This links the stability of the ground-state as a function of the filling fraction of the lowest Landau level with cooperative tunneling events between classically minimal energy configurations. The authors claim that this semiclassical picture is consistent with Laughlin's theory [4].

The tunnelling events identified by KKAS involve electrons tunnelling cooperatively around rings. The contributions to the partition function of these events add coherently at filling fraction of the lowest Landau level, $\nu = 1/(2p + 1)$, provided the (classically stable) initial and final configurations are those characteristic of an incompressible fluid. In any one process particles tend to remain well-separated and so at least for particles with short-range repulsive interactions these events should contribute to a lowering of the free energy.

KKAS found an approximate mapping between contributions from these ring exchange events to the partition function and those of domain walls in the discrete Gaussian model on a lattice. The order-disorder transition in the discrete Gaussian model corresponds to a transition between a 'sparse-ring' phase in which ring events do not make a significant contribution to the partition function and a 'dense-ring' phase. The 'sparse-ring' phase found to be stable at low filling fraction retains the essential features of the configurations coupled by the ring exchange events which at low enough temperatures should be Wigner-crystal-like. The 'dense-ring' phase is identified with the incompressible Laughlin fluid.

The transition between the two phases is calculated by KKAS to occur at a filling fraction of the lowest Landau level between $1/3$ and $1/5$. This is (perhaps remarkably given the level of approximation) consistent with recent experimental indications [5,6].

We consider here whether the semiclassical picture based on cooperative ring exchange (CRE) makes sense for spin-polarized electrons in a partially filled $n = 1$ Landau level. Although a partially occupied $n = 1$ Landau level may not be fully spin-polarized the properties of spin-polarized electrons in the $n = 1$ Landau level are known theoretically at some filling fractions from finite size studies [7]. In particular it is known that for filling fractions $\nu_1 < 1/3$ fractional quantum Hall states can be expected much as in the $n = 0$ Landau level for $\nu_0 < 1$. However, the Laughlin state at $\nu_1 = 1/3$ is not a good description of the ground state of electrons interacting via the Coulomb interaction in the $n = 1$ Landau level which is probably gapless and compressible [8].

We generalize the CRE description of the FQHE to the case of spin-polarized electrons in the $n = 1$ Landau level. The system is governed by one parameter, $\alpha(\nu)$, just as for the $n = 0$ case. $\alpha(\nu)$ measures the contribution per unit length of the ring of a single cooperative tunnelling process. We find that as in the $n = 1$ Landau level $\alpha(\nu)$ decreases monotonically with increasing filling fractions which was to be expected—at higher filling fractions the modulation of the particle density is less pronounced and the barriers to tunnelling are smaller.

We also address the question of how the compressibility of the ground state at $\nu_1 = 1/3$ might be reproduced in this picture. The phase transition in the discrete Gaussian model is identified with the Wigner crystal/Laughlin fluid transition. The transition incompressible Laughlin fluid/compressible fluid must have another origin [9]. We find that the Wigner crystal becomes unstable classically at a filling fraction $\nu_1 = 1/3$ at which the shear modulus vanishes. As the Wigner crystal is supposed to be the configuration with the lowest energy classically and forms the starting point for the semiclassical treatment it is clear that the treatment must break down before this point.

In the next section we present the generalization of KKAS to the $n = 1$ Landau level. In section 3 we give our results and section 4 is a summary. There are also three appendices which summarize the technical details of the generalizations of various results known for the $n = 0$ case to the $n = 1$ case.

2. Semiclassical formulation

The semiclassical calculation due to KKAS [3] assumes that spin-polarized electrons occupy a partially filled Landau level. The classical minimum energy state in the basis of coherent (most localized) states is that in which electrons occupy orbitals localized about the lattice points (written as complex numbers) $R_i = X_i + iY_i$ characteristic of the hexagonal Wigner-crystal lattice (see appendix A)

$$|R_1, R_2, \dots, R_N\rangle_n \quad (1)$$

The R_i are actually complex eigenvalues of the guiding centre operators b_i , defined in equation (10).

The difference between the calculation for the $n = 0$ and $n = 1$ Landau levels comes about as a result of the difference between the coherent states in the different Landau levels (see equation (12)). This gives rise to a different effective interaction between particles, $V_n(|R_1 - R_2|)$, see equations (19) and (20), but otherwise the expression for the $n = 1$ Landau level is as for the $n = 0$ Landau level.

The contribution to the partition function of a single tunnelling event of length L is approximated by (see appendix B)

$$\tilde{D}e^{-\tilde{S}} \approx \frac{d\tau}{\tau_0} e^{-\alpha_n(\nu)L}. \quad (2)$$

The term $\tilde{D}[R^c]$ accounts for the integral over fluctuations about the classical path R^c . The classical action is $\tilde{S}[R^c]$, the tunnelling time is τ_0 .

The sum over contributions from different tunnelling events is approximately equal to the partition function of the classical discrete Gaussian model in which α is the inverse temperature. This model has a phase transition which KKAS identified with the transition from Wigner crystal or 'sparse-ring' phase and Laughlin or 'dense-ring' phase. The critical value of α is estimated for the hexagonal lattice to lie between 0.7 and 1.3.

The values found for $\alpha_n(\nu)$ are given in table 1. We also list the coefficients Q_x , Q_y and κ which characterize the variation of the effective interaction between particles (see appendix B). These are all smaller at any given filling fraction in the $n = 1$ than in the $n = 0$ Landau level. This was to be expected as the less localized nature of the orbitals, $|R\rangle_{n=1}$, means that the density is more homogeneous and hence that the variations in the potential energy surface for the motion of electrons in the orbitals are less. The less localized nature of the orbitals also means that the effective mass in the action for the one-dimensional motion after integrating out the fluctuations perpendicular to the tunnelling path is also larger (it varies as $1/Q_y$) [3].

Table 1. The parametrization of the potential and action for a single cooperative tunnelling event along a straight line path in a hexagonal Wigner crystal, with lattice parameter a , for different filling fractions ν in the $n = 0$ and $n = 1$ Landau levels. The action of a single event is α per unit length (see equation (39)). α_0 , (equation (36)), is the classical action per unit length. The quantities Q_x , Q_y and κ parametrize the interparticle interaction about the straight line path, (see equation (34)).

		$Q_x[\frac{e^2}{\epsilon a}]$	$Q_y[\frac{e^2}{\epsilon a}]$	$\kappa[\frac{e^2}{\epsilon a}]$	α_0	α
$\nu = 1/3$	$n = 0$	1.2	11.1	3.5	1.47	1.28
	$n = 1$	0.62	11.0	3.5	1.05	0.88
$\nu = 1/5$	$n = 0$	0.99	9.5	2.9	2.37	2.16
	$n = 1$	1.53	12.5	4.0	2.57	2.42
$\nu = 1/7$	$n = 0$	0.87	8.9	2.7	3.22	3.0
	$n = 1$	1.2	10.6	3.3	3.46	3.27
$\nu = 1/9$	$n = 0$	0.82	8.6	2.6	4.1	3.9
	$n = 1$	1.0	9.6	3.0	4.3	4.1
$\nu = 1/11$	$n = 0$	0.79	8.5	2.6	5.0	4.7
	$n = 1$	0.92	9.2	2.8	5.1	4.9
$\nu = 1/13$	$n = 0$	0.78	8.4	2.5	5.8	5.6
	$n = 1$	0.89	9.0	2.7	6.0	5.8

The net result for the action of a single tunnelling event, $\alpha(\nu)L$, is that it does not differ greatly between the $n = 0$ and $n = 1$ Landau levels for $\nu = 1/5, 1/7, \dots$. The values for α in both Landau levels are greater than the estimate of the critical value $\alpha_c \sim 1.3$ so that our estimates imply that the critical filling fraction for the transition from Wigner crystal to Laughlin fluid occurs for filling fractions greater

than $1/5$, although given the number and quality of approximations involved this is at best only a rough prediction.

There is a larger difference between $\alpha_{n=1}(1/3)$ and $\alpha_{n=0}(1/3)$, with $\alpha_1 < \alpha_0$. According to the identification made by KKAS of the dense ring phase with the incompressible Laughlin state the smaller value for α in the $n = 1$ Landau level would imply that the $1/3$ state were more stable contradicting the results of finite size calculations. (These show that the $\nu_1 = 1/3$ Laughlin state is only marginally stable.) We show in the next section the $\nu_1 = 1/3$ result is in fact meaningless as the basis for the semiclassical approximation breaks down.

3. Classical stability of the Wigner crystal

The starting point for the semiclassical treatment has been the configurations which minimize the classical action. We have assumed that the Wigner crystal gives such a minimum. The test of the classical stability of the Wigner crystal is the shear modulus, c_1 . A negative shear modulus would imply that the Wigner crystal is not a stable configuration. In this section we report the calculation of the shear modulus.

Maki and Zotos [10] found that the calculation of the shear modulus requires the correct treatment of the antisymmetrization of the basis states (22). In appendix C we outline the definition and calculation of the shear modulus. We show the result in figure 1.

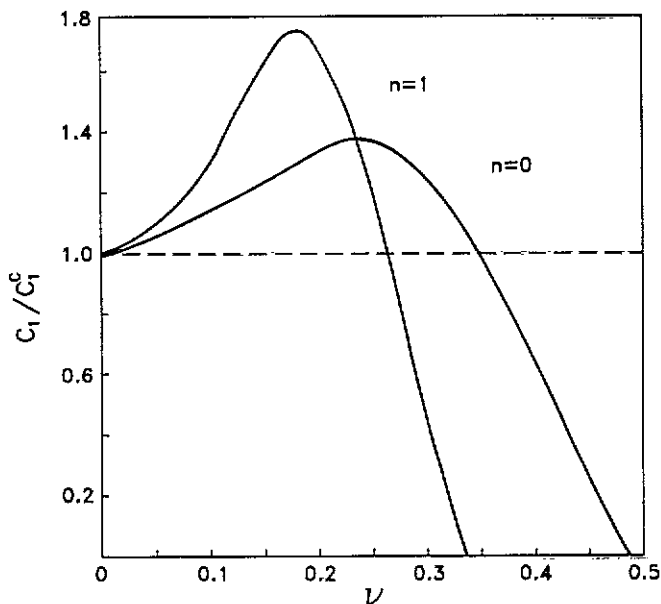


Figure 1. The shear modulus of the Wigner-crystal lattice in the $n = 0$ and $n = 1$ Landau levels as a function of filling fraction. The shear modulus is normalized to its classical value.

In the $n = 1$ Landau level the shear modulus vanishes very close to the filling fraction $\nu_1 = 1/3$. This result is only approximate as in the calculation we have neglected three- and higher-body terms in the effective interaction. The correction terms may well shift slightly the filling fraction at which the shear modulus vanishes, but it is unlikely to affect the main result, namely that at $\nu_1 = 1/3$ the basis for the semiclassical treatment of the quantum Hall effect is not given.

The apparent compressibility of the ground state of a system of polarized electrons in the $n = 1$ Landau level suggested by the results of finite size studies [7,8] is consistent with our result that the semiclassical treatment becomes invalid at $\nu_1 = 1/3$. Experimentally something like a plateau has been reported [11] but, as we have not taken account of inter-Landau level effects nor of the possibility that the systems studied experimentally are not fully spin-polarized, a direct comparison with experiment is not possible.

4. Summary

We have generalized the semiclassical treatment of KKAS to the $n = 1$ Landau level. The tunnelling probability, $\alpha_1(\nu)$, at the level of approximation we have used is not significantly different from $\alpha_0(\nu)$ for filling fractions $\nu < 1/3$ at which Laughlin incompressible fluids might be expected to be stable. This is the result of two effects which cancel: The tunnelling barriers are smaller owing to the reduced variation in charge density which results because the electrons cannot be so well localized in the $n = 1$ Landau level. The more delocalized nature of the electrons also leads to a larger effective mass in the action functional. The combination of these two effects leads to similar values for α in the two Landau levels.

The shear modulus of the Wigner crystal vanishes at $\nu_1 \approx 1/3$. The basis of the semiclassical treatment is therefore missing for filling fractions near $1/3$. This result is consistent with the result suggested by calculations for small systems of particles interacting via Coulomb interaction projected onto the $n = 1$ Landau level that the ground state at $\nu_1 = 1/3$ is compressible.

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Appendix A. The effective potentials

In the following we keep closely to the notation of KKAS. We consider electrons ($q = -e$) in a magnetic field $B = -B\hat{z}$ and work always in the symmetric gauge with the vector potential $A_i = \frac{1}{2}B\epsilon_{ij}r_j$. The magnetic length $l = \sqrt{\hbar/eB}$.

A.1. Cyclotron and guiding centre motions

It is helpful to separate the cyclotron and guiding centre motions of electrons. The cyclotron position, ξ_i , is

$$\xi_i = \frac{r_i}{2} + \frac{q l^2}{e \hbar} \epsilon_{ij} p_j \quad (3)$$

and the guiding centre position is given by

$$\hat{R}_i = \frac{r_i}{2} - \frac{q l^2}{e \hbar} \epsilon_{ij} p_j. \quad (4)$$

(The hat distinguishes the operator \hat{R} from the eigenvalue R , equation (11).)

Ladder operators, a (a^+), for cyclotron motion may be defined

$$a = [\xi_x - i(q/e)\xi_y]/\sqrt{2}l. \quad (5)$$

These lower (raise) the Landau level index, n . Raising and lowering operators for the angular momentum about the origin, b and b^+ , may also be defined:

$$b = [\hat{R}_x + i(q/e)\hat{R}_y]/\sqrt{2}l. \quad (6)$$

These operators obey the canonical commutation relations $[a, a^+] = [b, b^+] = 1$.

A.2. Coherent states

The basis states for our calculations are the coherent states localized about the points $R = X + iY$

$$|R\rangle_n = \frac{(a^+)^n}{n!} e^{(b^+ - Rb)/\sqrt{2}l} |0\rangle_0 \quad (7)$$

where $|0\rangle_0$ satisfies

$$a|0\rangle_0 = 0 \quad (8)$$

$$b|0\rangle_0 = 0. \quad (9)$$

The states $|R\rangle_n$ are eigenstates of b

$$b|R\rangle_n = \frac{\bar{R}}{\sqrt{2}l} |R\rangle_n \quad (10)$$

$${}_n\langle r|b^+ = {}_n\langle R|\frac{R}{\sqrt{2}l}. \quad (11)$$

Here $\bar{R} = X - iY$. The corresponding Schrödinger wavefunctions are

$$\phi_{R,n} = \frac{1}{(2\pi l^2 n!)^{1/2}} \left(\frac{\bar{z} - \bar{R}}{\sqrt{2}l} \right)^n e^{(z\bar{R} - \bar{z}R)/4l^2} e^{-|z-R|^2/4l^2}. \quad (12)$$

These are not orthogonal as

$${}_{n'}\langle R' | R \rangle_n = \delta_{nn'} e^{(R'\bar{R} - R\bar{R}')/4l^2} e^{-|R-R'|^2/4l^2}. \quad (13)$$

The N -body states we use as a basis in the estimation of the ring exchange action are just the unsymmetrized product states

$$\psi = |R_1, R_2, \dots, R_N\rangle_n. \quad (14)$$

Particles occupy coherent states at the guiding centre coordinates R_1, \dots, R_N . In the classical minimum energy configuration these form the lattice points of a 2D hexagonal lattice.

A.3. Matrix elements

The matrix elements of an arbitrary function, $f(r_{12})$, of the separation between two particles, r_{12} , between states ψ equation (14) are more easily evaluated if we write [8]:

$$\begin{aligned} f(r_{12}) &= \int \frac{d^2 Q}{2\pi} \tilde{f}(Q) e^{iQ \cdot r_{12}} = \int \frac{d^2 Q}{2\pi} \tilde{f}(Q) e^{iQ \cdot (\hat{R}_{12} + \xi_{12})} \\ &= \int \frac{d^2 Q}{2\pi} \tilde{f}(Q) e^{-Q^2 l^2 / 2} e^{iQ \cdot \hat{R}_{12}} e^{i\hat{Q} l a_{12}^+ / \sqrt{2}} e^{iQ l a_{12} / \sqrt{2}}. \end{aligned} \quad (15)$$

Taken between states which have both particles in the n th Landau $f(r_{12})$ may be replaced by:

$$\int \frac{d^2 Q}{2\pi} \tilde{f}(Q) e^{-Q^2 l^2 / 2} (L_n(Q^2 l^2 / 2))^2 e^{iQ \cdot \hat{R}_{12}} \quad (16)$$

and after writing the operator $e^{iQ \cdot \hat{R}_{12}}$ as:

$$e^{iQ \cdot \hat{R}_{12}} = e^{i\hat{Q} l b_{12}^+ / \sqrt{2}} e^{-Q^2 l^2 / 2} e^{iQ l b_{12} / \sqrt{2}} \quad (17)$$

the evaluation of matrix elements is straightforward.

A.4. Effective two-body potential

In the estimation of the contribution from the Coulomb interaction between electrons to the tunnelling action we retain only the two-body interaction (in the following we set $l = 1$):

$$\begin{aligned} V(R_{12}) &= {}_n \langle R_1, R_2 | 1 / (|r_1 - r_2|) | R_1, R_2 \rangle_n \\ &= \int \frac{d^2 Q}{2\pi} \frac{1}{Q} (L_n(Q^2 / 2))^2 e^{iQ \cdot (R_1 - R_2)} e^{-Q^2} \end{aligned} \quad (18)$$

where we have used equations (16) and (17). This gives for $n = 0$ and $n = 1$

$$V_{n=0} = \frac{e^2 \sqrt{\pi}}{2\epsilon} e^{-\alpha} I_0(\alpha) \quad (19)$$

$$V_{n=1} = \frac{e^2 \sqrt{\pi}}{2\epsilon} e^{-\alpha} \frac{1}{16} [I_0(\alpha)(11 + 4\alpha + 8\alpha^2) - I_1(\alpha)(8\alpha + 8\alpha^2)] \quad (20)$$

where $\alpha = R_{12}^2 / 8$ and $I_{0(1)}$ are modified Bessel functions. The derivation of equation (20) follows simply after writing

$$(L_1(Q^2 / 2))^2 e^{-Q^2} = (1 - Q^2 + Q^2 / 4) e^{-Q^2} = \left(1 + \frac{d}{d\lambda} + \frac{1}{4} \frac{d^2}{d\lambda^2} \right) e^{-\lambda Q^2} \Big|_{\lambda=1}. \quad (21)$$

A.5. Maki-Zotos potential

In the calculation of the shear modulus Maki and Zotos [10] found that it was necessary to assume fully antisymmetrized basis functions

$$|\psi\rangle = (N!)^{-\frac{1}{2}} \sum_{\sigma \in S_N} (-)^{\sigma} |R_{\sigma(1)}, R_{\sigma(2)}, \dots, R_{\sigma(N)}\rangle_n \quad (22)$$

where S_N is the set of permutations of N objects. $|\psi\rangle$ is just a Slater determinant of single-particle coherent states. The expectation of the interaction energy as a function of the positions R_1, \dots, R_N may be expanded into effective two-body and higher order terms

$$E = \frac{e^2}{\epsilon} \langle \psi | \sum_{j < k} \frac{1}{|r_{jk}|} |\psi\rangle / \langle \psi | \psi \rangle \\ = \sum_{j < k} V^{MZ}(R_{jk}) + \sum_{j < k < l} V_3(R_j, R_k, R_l) + \dots \quad (23)$$

As Maki and Zotos we retain only the effective two-body interaction V^{MZ} , which is given by

$$\frac{\epsilon}{e^2} V^{MZ}(R_{12}) = \frac{n \langle R_1, R_2 | 1/r_{12} | R_1, R_2 \rangle_n - n \langle R_1, R_2 | 1/r_{12} | R_2, R_1 \rangle_n}{1 - n \langle R_1, R_2 | R_2, R_1 \rangle_n}. \quad (24)$$

For the $n = 0$ and $n = 1$ Landau levels this gives

$$V_{n=0}^{MZ} = \frac{\sqrt{\pi} e^2}{4\epsilon} \operatorname{sech} \alpha I_0(\alpha) \quad (25)$$

$$V_{n=1}^{MZ} = \frac{\sqrt{\pi} e^2}{4\epsilon} \frac{1}{16} [(I_0(\alpha)(11 + 8\alpha^2) - I_1(\alpha)8\alpha) \operatorname{sech} \alpha \\ + (I_0(\alpha)4\alpha - I_1(\alpha)8\alpha^2) \operatorname{cosech} \alpha] \quad (26)$$

with $\alpha = R^2/8$. The result (25) was first reported by Maki and Zotos [10].

Appendix B

The partition function in the coherent state basis is (assuming continuous paths) [3]

$$Z(\nu) \sim N \sum_{\sigma \in S_N} \operatorname{sign}(\sigma) \int \prod_{j=1}^N \mathcal{D}R_j(\tau) e^{-S(\{R(\tau)\})} \quad (27)$$

with

$$S(\{R(\tau)\}) = \int_0^\beta d\tau \left[\sum_{j=1}^N \left(-\frac{i}{2} \dot{R}_j \wedge R_j + U(R_j) \right) + \sum_{j < k} V_n(R_j - R_k) \right]. \quad (28)$$

Here the $R_j(\tau) = (X_j(\tau), Y_j(\tau))$ label the coherent states occupied by particles j at imaginary time τ and satisfy

$$R_j(0) = R_{\sigma(j)}(\beta) \quad (29)$$

The sum is over all permutations, σ , of the N particles.

The semiclassical approximation consists of identifying the extremal paths of the action (28), $R^c(\tau)$, with corresponding actions $S(R^c)$, and integrating over quantum fluctuations about these paths. The approximate evaluation of these integrals involves expanding the action up to quadratic order in the deviation from the classical path. Assuming that the paths are independent and do not interfere, the partition function may be written as a sum over the classical extremizing paths

$$Z = \sum_c e^{-S(R^c)} D[R^c]. \quad (30)$$

$D[R^c]$ is called the fluctuation determinant and accounts for the contributions from the fluctuations about R^c . The continuous classical extremal paths satisfy

$$\begin{aligned} i\dot{X}_j &= \frac{\partial V}{\partial Y_j} \\ i\dot{Y}_j &= -\frac{\partial V}{\partial X_j}. \end{aligned} \quad (31)$$

As KKAS we assume that the contributions to Z may be counted by starting from the classically minimal energy state the Wigner crystal with corresponding action, S_0 . Baskaran [12] has argued that ring exchange events between any local classical energy minima which might be found in an incompressible fluid will contribute to the non-analytic behaviour of $Z(\nu)$ at $\nu = 1/(2n + 1)$. However the contribution from these additional configurations is thought to provide just a (probably small) renormalization to the non-analyticity [3].

The contribution to Z/Z_0 of a single ring exchange event involving L particles is given approximately by

$$\bar{D}e^{-\tilde{S}(R^c)} = \frac{d\tau}{\tau_0} \exp[-\alpha_n(\nu)L \pm i2\pi f N_A + O(\ln L)] \quad (32)$$

where

$$f = \frac{1}{2}(\nu^{-1} - 1). \quad (33)$$

The factor Z_0 has been divided out so that we need only concentrate on the difference $\tilde{S}(R^c) = S - S_0$ between the action of the exchange event and the action of the Wigner crystal. N_A is the number of plaquettes of the triangulated classical configurations between which the system is tunnelling and which are enclosed by the ring. The factor $d\tau/\tau_0$ is the usual factor in instanton calculations in which an integration over the event time is included. τ_0 is the classical time for the tunnelling event to occur.

Equation (32) follows after assuming that the action of the event does not depend on the shape of the ring so that the contribution is determined uniquely by its length. On the basis of this assumption all the effects of the different basis states for the different Landau levels are accounted for by the different values taken by $\alpha_n(\nu)$ in the different Landau levels labelled by the index n .

B.1. Calculation of $\alpha_n(\nu)$

The treatment of ring exchange in the $n = 1$ Landau level reduces in the approximation of KKAS to the estimation of $\alpha_{n=1}(\nu)$. We assume that the effects of variations of the total area of a path associated with deviations from straight line paths particularly near corners is, as found in the $n = 0$ Landau level, just to give a small renormalization of the quantity α . (We we have not checked this numerically for the $n = 1$ Landau level.) We therefore consider just a simple 'straight line exchange' path.

For cooperative tunnelling of particles along a straight line path of length, L , parallel to the x direction the interaction term in the action integral is approximated by

$$V = \frac{e^2}{\epsilon a_\nu} \left\{ \sum_{j=1}^L \frac{Q_y}{2} \left[\frac{Y_j}{a_\nu} \right]^2 + \frac{Q_x}{(2\pi)^2} [1 - \cos 2\pi X_j / a_\nu] \right. \\ \left. + \sum_{k < j} \frac{1}{2} \left(K_x(k-j) \left[\frac{X_k - X_j}{a_\nu} \right]^2 + K_y(k-j) \left[\frac{Y_k - Y_j}{a_\nu} \right]^2 \right) \right\}. \quad (34)$$

The constants Q_x and Q_y are evaluated numerically by fitting to the expansion of the true potential V_n . Integrating out the motion in the y direction gives an effective one-dimensional action functional [3]. The classical action is then

$$\tilde{S}[R^c] = \alpha_0(\nu)L \quad (35)$$

with

$$\alpha_0(\nu) = (Q_x/Q_y)^{1/2} (8/\sqrt{3}\pi) \nu^{-1}. \quad (36)$$

The fluctuation determinant $\tilde{D}(R^c)$ is evaluated approximately by mapping the action functional to that of a sine-Gordon field, $\phi(x)$. As mentioned in KKAS the usual procedure for taking the continuum limit would make the replacement

$$\frac{1}{2} \sum_{k < j} K_x(k-j) \left[\frac{X_j - X_k}{a_\nu} \right]^2 \rightarrow \int_0^L \frac{1}{2} \frac{\kappa}{(2\pi)^2} \phi_x^2$$

where $\phi(j) = 2\pi X_j / a_\nu$. This procedure needs care as κ cannot be set equal to the sum $\sum_j \frac{1}{2} (j a_\nu)^2 K_x(j)$ which is logarithmically divergent. Instead we evaluate the parameter κ numerically via

$$\kappa = \left. \frac{\tilde{K}_x(0) - \tilde{K}_x(k)}{k^2} \right|_{k=(k_Q)} \quad (37)$$

where $\tilde{K}(k)$ is the Fourier coefficient in the expansion

$$K_x(j-l) = \frac{1}{L} \sum_k \tilde{K}_x(k) e^{-ik(j-l)},$$

$k_Q = \sqrt{\kappa/Q_x}$ is a typical wavelength at which the kinetic and potential contributions to the action are about equal.

The integration over the sine-Gordon field is known [13]. The final result for the contribution to the effective action for a straight line tunnelling process involving L sites is

$$\tilde{D}e^{-\tilde{S}} \approx \frac{d\tau}{\tau_0} e^{-\alpha(\nu)L} \quad (38)$$

where $\alpha(\nu)$ is given by

$$\alpha(\nu) = \nu^{-1} \left(\frac{8}{\sqrt{3}\pi} \right) \left(\frac{Q_x}{Q_y} \right)^{1/2} \left(\frac{Q_x}{4\kappa} \right)^{g/16\pi} \left(1 - \frac{g}{8\pi} \right) \quad (39)$$

Here

$$\frac{g}{8\pi} = \left(\frac{Q_y}{\kappa} \right)^{1/2} \left(\frac{\sqrt{3}}{8} \right) \nu. \quad (40)$$

Expression (39) depends on the three parameters Q_x , Q_y and κ which characterize the interaction between the particles. In table 1 we give the results for the $n = 0$ and $n = 1$ Landau levels for the parameters Q_x , Q_y , κ and α for various ν .

Appendix C. The shear modulus

The shear modulus, c_1 , measures the stiffness (speed of sound) of a lattice. A negative shear modulus indicates that the lattice is unstable. Both KKAS and Maki and Zotos have given expressions for c_1 for general two-body interactions $V(R_{12})$ between lattice sites in a hexagonal lattice

$$c_1 = \frac{1}{16} \sum_R (3RV' + R^2V'') \quad (41)$$

where the sum is over lattice positions. This can also be written as a sum over reciprocal lattice vectors

$$c_1 = \frac{1}{16\Omega_\nu} \sum_{G \neq 0} (3\tilde{V}'(G)G + \tilde{V}''(G)G^2) \quad (42)$$

where $\tilde{V}(G)$ is the Fourier transform of $V(R)$.

Neither expression for c_1 converges rapidly as, for large separations, the effective interaction $V_n^{MZ}(R)$ (equations (25) and (26)) is asymptotic to $1/R$. However, if we separate out the contribution from the Coulomb interaction and write

$$c_1 = c_1^c + \sum_R 3R(V'(R) - V'_{\text{Coul}}) + R^2(V''(R) - V''_{\text{Coul}}) \quad (43)$$

then with $V(R) = V_n^{MZ}(R)$ the sum converges quite rapidly. c_1^2 has been calculated by Bonsall and Maradudin and is known for the hexagonal lattice to be $0.09775\nu^{1/2}e^2/l$.

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